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J-symmetric canonical models

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On a Hilbert space K to be specified below, we consider a bounded operator J such that $J=J^*=J^{-1}$. This implies there exist two orthogonal projections P_+ and P_- for which $I=P_++P_-$, $J=P_+-P_-$, and $P_+P_-=0$. Hence, we can write $K=K_+\oplus K_-$, where $K_\pm=P_\pm K=\{x\in K|Jx=\pm x\}$. A bounded operator A is called J-symmetric iff $A=JA^*J$. These operators have been widely studied and [3, 4] give references to the literature. Recently, P. A. FUHRMANN [2] characterized the J-symmetric restricted shifts T_φ acting on $(\varphi H^2)^\perp$, where φ is a scalar inner function, as those generated by φ having real Taylor coefficients. In this note, we extend Fuhrmann's results to a more general class of operators which have applications in linear systems theory.

Let C and C_* be separable Hilbert spaces and let $L^2(C)$, $L^2(C_*)$, $H^2(C)$, and $H^2(C_*)$ denote the standard vector-valued Lebesgue and Hardy spaces defined on the unit circle. (See [6] for a general reference.) We use "t" to denote the argument of a function defined on the unit circle, and for analytic functions (vector or operator valued), we freely identify $h(t)$ on the circle with $h(z)$, its extension to the disc. Let φ denote a fixed purely contractive analytic operator-valued function from C to C_*, i.e. $\varphi(z)\colon C\to C_*$ with $\|\varphi(z)\|\leqq 1$, $\varphi(z)c\in H^2(C_*)$ for all $c\in C$, and $\|\varphi(0)c\|<\|c\|$ for all $c\in C$, $c\neq 0$. Let $\Delta(t)=(I-\varphi(t)^*\varphi(t))^{1/2}$ and let $H=H^2(C_*)\oplus\overline{\Delta L^2(C)}$. Then $M=\{(\varphi(z)f(z), \Delta(t)f(t))|f\in H^2(C)\}$[1]) is invariant under U_+, the unilateral shift on H defined by $U_+(f,g)=(zf, e^{it}g)$, so $K=H\ominus M$ is invariant under U_+^*. Let P denote the projection of H onto K, and let T be the compression of U_+ onto K; thus, $T(f,g)=P(zf, e^{it}g)$ for $(f,g)\in K$. In this context, K is called the Sz.-Nagy—Foiaş space generated by φ, and T is called a canonical model. The Sz.-Nagy—Foiaş model theorem states that any completely non-unitary contraction S is unitarily equivalent to the canonical model on the space generated by a contractive operator-valued analytic function which coincides with the characteristic function of S [6, Chapter VI].

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[1]) Since we shall not use inner products in this paper, we write (f, g) for $f\oplus g$.

($\varphi_1(z)$ and $\varphi_2(z)$ coincide iff $\varphi_1(z)=A\varphi_2(z)B$ for some constant unitary A and B; characteristic functions are necessarily purely contractive [6, p. 239].) Note that if φ is inner,[2]) i.e. $\varphi(t)$ is unitary a.e., then $\Delta(t)=0$ a.e. so $H=H^2(C_*)\ominus\varphi H^2(C)$, and if $\dim C=\dim C_*=1$, then $\varphi(z)$ is a scalar-valued function acting by multiplication, so restricted shifts are special cases of canonical models.

Given φ, define $\tilde{\varphi}(z)=\varphi(\bar{z})^*$, an analytic purely contractive function mapping C_* to C. Note that φ is inner iff $\tilde{\varphi}$ is inner. Analogously to above, let $\tilde{\Delta}(t)=(I-\tilde{\varphi}(t)^*\tilde{\varphi}(t))^{1/2}$, $\tilde{H}=H^2(C)\oplus\overline{\tilde{\Delta}L^2(C_*)}$, $\tilde{K}=\tilde{H}\ominus\{(\tilde{\varphi}f,\tilde{\Delta}f)|f\in H^2(C_*)\}$, and $\tilde{T}(f,g)=\tilde{P}(zf,e^{it}g)$ for $(f,g)\in\tilde{K}$, where $\tilde{P}$ projects $\tilde{H}$ onto $\tilde{K}$. We define τ on K by

$$(1)\qquad \tau(f,g)=e^{-it}(\varphi(-t)^*f(-t)+\Delta(-t)g(-t),\ \tilde{\Delta}(t)f(-t)-\varphi(-t)g(-t))),$$

one can show that τ is a unitary map of K onto $\tilde{K}$ for which $\tilde{T}\tau=\tau T^*$, and $\tau^{-1}=\tau^*=\tilde{\tau}$ mapping $\tilde{K}$ onto K is defined by a formula analogous to (1) for $\tilde{\varphi}$ in place of φ [1]. Thus, if $\varphi=\tilde{\varphi}$, then τ is a J-operator on K and T is J-symmetric. We see below that for scalar functions, $\varphi=\tilde{\varphi}$ is also necessary for T to be J-symmetric, provided we normalize φ by requiring its first non-vanishing Taylor coefficient to be positive. We get similar results in the vector case.

Before proceeding to the main theorem, we establish two lemmas. The first relies on the following theorem of B. Sz.-Nagy and C. Foiaş.

Theorem. (i) (The lifting theorem, [6, II. 2]). *Let T_i be the canonical model on $K_i\subset H_i$, $i=1,2$. If $V\colon K_1\to K_2$ such that $VT_1=T_2V$ (i.e. V intertwines T_1 and T_2), then $V=PY|K_1$ for some $Y\colon H_1\to H_2$ such that $U_{+_2}Y=YU_{+_1}$, $PYM=0$, and $\|Y\|=\|V\|$.*

(ii) [7, p. 235] *The map Y above has the form*

$$(Y(f,g))(t)=Y(t)\begin{bmatrix} f(t)\\ g(t)\end{bmatrix}\quad \textit{where}\quad Y(t)=\begin{bmatrix} A(t) & 0\\ B(t) & C(t)\end{bmatrix}$$

for some bounded analytic $A(t)\colon C_{*_1}\to C_{*_2}$ and bounded measurable $B(t)\colon C_{*_1}\to\overline{\Delta_2(t)C_2}$, $C(t)\colon\overline{\Delta_1(t)C_1}\to\overline{\Delta_2(t)C_2(t)}$ such that $A\varphi_1(t)=\varphi_2(t)A_*(t)$ and $B(t)\varphi_1(t)+C(t)\Delta_1(t)=\Delta_2(t)A_*(t)$ a.e., for some bounded analytic $A_*(t)\colon C_1\to C_2$.

Lemma 1. *$V\colon K_1\to K_2$ intertwining T_1 and T_2 is unitary if and only if $V=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$ for some unitary maps $\alpha\colon C_{*_1}\to C_{*_2}$, $\beta\colon C_1\to C_2$ such that*

$$\alpha\varphi_1(t)=\varphi_2(t)\beta\quad\textit{and}\quad \alpha^*\varphi_2(t)=\varphi_1(t)\beta^*\ \textit{a.e.}$$

Proof. We have $V=PY$ with Y as in the previous theorem, but since $\|Y\|=1$ and V is unitary, $Y=V$ in K_1. On K_2, $Y^*=V^*$ is also unitary so $Y^*=\begin{bmatrix} A(t)^* & B(t)^*\\ 0 & C(t)^*\end{bmatrix}$,

[2]) 'Inner from both sides' in the sense of [6], p. 190.

which implies $B(t)=0$ and $A(t)=\alpha$ is constant a.e. Clearly, $\alpha\colon C_{*_1}\to C_{*_2}$ is unitary and $C(t)\colon \overline{\Delta_1(t)L^2(C_1)}\to\overline{\Delta_2(t)L^2(C_2)}$ is unitary a.e., and by the theorem applied to Y and Y^*, we have

$$
(2)\qquad \begin{aligned} \alpha\varphi_1(t) &= \varphi_2\beta(t) \\ C(t)\Delta_1(t) &= \Delta_2(t)\beta(t) \end{aligned} \quad\text{and}\quad \begin{aligned} \alpha^*\varphi_2(t) &= \varphi_1(t)\gamma(t) \\ C(t)^*\Delta_2(t) &= \Delta_1(t)\gamma(t) \end{aligned} \quad \text{a.e.,}
$$

for some analytic $\beta(t)$, $\gamma(t)$. Using (2), we have

$$\varphi_1(t)^*\varphi_1(t) = (\varphi_1(t)^*\alpha^*)(\alpha\varphi_1(t)) = \beta(t)^*\varphi_2(t)^*\varphi_2(t)\beta(t)$$

and

$$\Delta_1(t)^2 = (I-\varphi_1(t)^*\varphi_1(t)) = (\Delta_1(t)C(t)^*)(C(t)\Delta_1(t)) = \beta^*(t)\Delta_2^2(t)\beta(t),$$

so $\beta(t)^*\beta(t)=I$ a.e. Similarly we see $\beta(t)\gamma(t)=I$ a.e. and hence $\gamma(t)=\beta(t)^{-1}=\beta(t)^*$ a.e. is analytic so $\beta(t)=\beta$ is constant a.e. Since $\beta=\gamma^*$, (2) yields $C(t)\Delta_1^2(t)=\Delta_2^2C(t)$, which implies that $C(t)\Delta_1(t)=\Delta_2(t)C(t)$ a.e. since Δ_i is a positive contraction. Consequently, $\beta C(t)^*=I$ on $\Delta_2L^2(C_2)$, so $C(t)=\beta$ a.e. The converse follows immediately. Note that if φ is a scalar function, then $\alpha=\beta$ is a complex number of modulus one and V is multiplication by a scalar.

Lemma 2. *For $|w|<1$, $x\in C_*$, $y\in C$, define*

$$d_{w,x} = \left(\frac{I-\varphi(z)\varphi(w)^*}{1-z\bar{w}}x,\ -\frac{\Delta(t)\varphi(w)^*}{1-e^{it}\bar{w}}x\right)$$

and

$$D_{w,y} = \left(\frac{\varphi(z)-\varphi(\bar{w})}{z-\bar{w}}y,\ -\frac{\Delta(t)}{e^{it}-\bar{w}}y\right).$$

Then $d_{w,x}$ and $D_{w,y}$ are in K and

(i) $d_{w,x} = P(x/(1-z\bar{w}),0)$ *and* $D_{w,y} = P(\varphi(t)y/(e^{it}-\bar{w}),\ \Delta(t)y/(e^{it}-\bar{w})$.

(ii) *if we define $\tilde{d}_{w,y}$ and $\tilde{D}_{w,x}$ analogously for $\tilde{\varphi}$, then*

$$\tau d_{w,x} = \tilde{D}_{w,x} \quad\text{and}\quad \tau D_{w,y} = \tilde{d}_{w,y}.$$

(iii) *For $F=(f,g)\in K$ and (τ_1F) the first coordinate of τF, $(F,d_{w,x})_K=(f(w),x)_{C*}$ and $(F,D_{w,y})_K=((\tau_1F)(w),y)_C$.*

(iv) *The linear span of $\{d_{w,x}+D_{w,y}\,\|\,|w|<1,\ x\in C_*,\ y\in C\}$ is dense in K.*

Proof. These all follow from straightforward computations and are found in [1]. The duality in (ii) is helpful for showing (iii) and (iv).

Theorem 1. *T is J-symmetric if and only if $\tilde{\varphi}(z)=A\varphi(z)A$ (i.e. $\tilde{\varphi}$ coincides with φ), where A is an arbitrary unitary map from C_* to C. In this case,*

$$J = \pm\begin{bmatrix} A^* & 0 \\ 0 & A \end{bmatrix}\tau.$$

Proof. If $T^*=JTJ$ for some J, then $V=\tau J$ is unitary and $\tilde{T}V=VT$. Thus by Lemma 1, $J=\tau^*\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$ with α, β unitary, $\alpha\varphi=\tilde{\varphi}\beta$, $\alpha^*\tilde{\varphi}=\varphi\beta^*$. Using these properties, we have

$$Jd_{w,x} = \tau^*\tilde{d}_{w,\alpha x} = D_{w,\alpha x};\quad JD_{w,y} = \tau^*\tilde{D}_{w,\beta y} = d_{w,\beta y};$$

$$J^*d_{w,x} = \begin{bmatrix}\alpha^* & 0\\ 0 & \beta^*\end{bmatrix}\tilde{D}_{w,x} = D_{w,\beta^* x};\quad \text{and}\quad J^*D_{w,y} = \begin{bmatrix}\alpha^* & 0\\ 0 & \beta^*\end{bmatrix}\tilde{d}_{w,y} = d_{w,\alpha^* y}.$$

Since $J=J^*$, we have $\alpha=\beta^*$, so $\alpha\varphi=\tilde{\varphi}\alpha^*$, and $\alpha\varphi\alpha=\tilde{\varphi}$.

Conversely, if $\tilde{\varphi}=A\varphi A$, $\pm\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tau$ is a J-operator since $\{d_{w,x}+D_{w,y}\}$ spans K. Using this J, T is symmetric.

In the scalar case, α and β are complex numbers of modulus one. If we normalize φ by requiring the first nonvanishing Taylor coefficient to be positive, then $\alpha^2=1$ and we have the following

Corollary 1. *If T is a scalar canonical model, i.e., $\varphi(z)$ is a (normalized) scalar function, then T is J-symmetric if and only if all the Taylor coefficients of φ are real, and $J=\pm\tau$.*

Theorem 2. *Let $\tilde{\varphi}=A\varphi A$ as in Theorem 1, so T is J-symmetric for $J=\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tau$. Let K_+ and K_- be defined by*

$$K_\pm = \text{closed span } \{d_{w,x}\pm D_{w,Ax} \mid x\in C_*, |w|<1\}.$$

Then $K_\pm = \{f\in K \mid Jf=\pm f\}$.

Proof. $Jd_{w,x} = \begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tilde{D}_{w,x} =$

$$= \left(A^*(z-\bar{w})^{-1}(\tilde{\varphi}(z)-\tilde{\varphi}(\bar{w}))x,\ A(e^{it}-\bar{w})^{-1}\tilde{\Delta}(t)x\right) =$$

$$= \left((z-w)^{-1}(\varphi(z)-\varphi(\bar{w}))Ax,\ (e^{it}-\bar{w})^{-1}\tilde{\Delta}(t)Ax\right) = D_{w,Ax}.$$

Similarly, $JD_{w,y}=d_{w,A^*y}$, so $J=\pm I$ on $K_\pm$. The subspaces are clearly orthogonal since $F\in K_+$, $G\in K_-$ implies $(F,G)=(JF,G)=(F,JG)=-(F,G)$, and by lemma 2, $K_+\oplus K_-$ spans K.

Corollary 2. *If* $\dim C<\infty$, *then K_+ is finite dimensional if and only if $\varphi(z)$ is of finite Blaschke type. The same holds for K.*

Proof. We note that if $\dim C=n$, then $\dim C_*=n$ since φ and $\tilde{\varphi}$ coincide, and $\varphi(z)$ can be realized as an $n\times n$ matrix whose entries are scalar H^∞ functions. We say φ is of finite Blaschke type iff $\det(\varphi(z))$ is a finite Blaschke product. Alternatively, the structure of contractive functions of finite-dimensional spaces is described in great detail in [5]; in that context the terminology is self-evident.

If $\varphi(z)$ is of finite Blaschke type, then K is finite-dimensional, and thus so are K_+ and K_-. Conversely, suppose $\dim(K_+)=N<\infty$. Since the second coordinate of $(d_{w,x}+D_{w,Ax})$ is $\Delta(t)\left(-(1-e^{it}\overline{w})^{-1}\varphi(w)^*x+e^{-it}(1-e^{-it}\overline{w})^{-1}Ax\right)$, it follows that $\Delta(t)=0$ a.e., so φ must be inner. (Note that this is still true if $\dim C=\infty$.) For w_j, $j=1, \dots, N+1$ distinct points in D, there exist constants a_j such that

$$\sum_{j=1}^{N+1} a_j(d_{w_j,x}+D_{w_j,Ax}) = 0.$$

Rearranging terms yields

$$\varphi(z)p(z)x = q(z)x, \quad \text{where} \quad p(z) = \sum_{j=1}^{N+1} a_j((1-z\overline{w}_j)^{-1}\varphi(w_j)^*-(z-\overline{w}_j)^{-1}A)$$

and

$$q(z) = \sum_{j=1}^{N+1} a_j((1-z\overline{w}_j)^{-1}I-(z-\overline{w}_j)^{-1}\varphi(\overline{w}_j)A).$$

Taking determinants shows that $\det(\varphi(z))$ is a rational function, so $\varphi(z)$ is of finite Blaschke type. A similar argument holds for K_-.

If [5, p. 212], $\varphi(z)=B(z)D$ where $B(z)$ is a diagonal matrix whose jth entry is $b_j(z)$, a scalar finite Blaschke product, and D is a constant unitary matrix, we can normalize $B(z)$ by requiring that each component be normalized in the scalar sense. Recall we have $\tilde{B}(z)=AB(z)A$; it is now easy to see that if the $b_j(z)$ are distinct, then (A) must be a diagonal matrix with entries ± 1 on the diagonal. If some $b_j(z)$ coincide, then (A) can be a block diagonal matrix, with blocks corresponding to coinciding $b_j(z)$, and each diagonal block a J-matrix. In any case, we have $\tilde{B}(z)=$ $=B(z)$, so we may take $J=\tau_B$ in theorem 2, where τ_B: $[BH^2(C)]^\perp\to[BH^2(C)]^\perp$. Clearly,

$$[BH^2(C)]^\perp = \oplus\sum_{j=1}^{N}(b_jH^2)^\perp, \quad \tau_B = \oplus\sum_{j=1}^{N}\tau_{b_j}, \quad \text{and} \quad K_\pm = \oplus\sum_{j=1}^{N}(K_\pm)_j.$$

Fuhrmann showed [2] that $\dim(K_+)_j=\left[\frac{n_j+1}{2}\right]$ and $\dim(K_-)_j=\left[\frac{n_j}{2}\right]$, where n_j is the number of factors in $b_j(z)$, and "[ ]" denotes the greatest integer function. Thus, we have determined the signature of $K_\pm$ in this special case. In general with $\dim C<\infty$, a finite Blaschke type inner function has the representation

$$B(z) = \prod_{k=1}^{n} B_k(z)U_k, \text{ where } B_k(z) = \begin{bmatrix} I_1 & 0 & \\ 0 & b_k(z) & I_2 \end{bmatrix}, \; b_k(z) = (z_k-a_k)(1-\bar{a}_kz)^{-1},$$

I_1 and I_2 are appropriate identity matrices, and U_k is a constant unitary matrix [5]. In this case, the signature is more difficult to determine. If $\dim C=\infty$, then $\dim(K)=\infty$ so either $\dim(K_+)$ or $\dim(K_-)$ (and in fact usually both) will be infinite. However, if $\varphi(z)$ is an infinite diagonal matrix whose first entry is a finite Blaschke product and all of whose remaining diagonal entries are $(z-\lambda)(1-\lambda z)^{-1}$, $-1<\lambda<1$, then we see $\dim(K_+)=\infty$ and $\dim(K_-)$ can be finite.

Corollary 3. *Let φ be a contractive operator-valued function. Then T on K is self-adjoint if and only if*

$$\varphi(z) = A^*(z + A\varphi(0))(I + z\varphi(0)^*A^*)^{-1},$$

where A is an arbitrary unitary matrix such that $A\varphi(0) = \varphi(0)^*A^*$.

Proof. If T is self-adjoint, then it is J-symmetric for $J=I$. The corollary follows from the computations in the proof of Theorem 1. Note that $\varphi(z)$ above is inner.

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